Announcements

1) HW 2 up later today

Recall:

Definition: (cardinality, countability)

Two sets S and T have the same cardinality if $\exists$ a bijection $f: S \to T$.

S is said to be countable if S has the cardinalility of the natural numbers

Example 1. a) 2 finite sets have the same cardinality if and only if they have the same number of elements

b) Any finite set has strictly smaller cardinality than any infinite set.

Question: Are infinite sets "stratified" by cardinality?

i.e. 1) If $S \subsetneq T$ and both S and T have infinite cardinality, must the cardinality of T be "greater" than the cardinality of S?

2) If the cardinality of T is infinite, is there a set S whose cardinality is strictly greater than that of T?

Notation: $|S|$ = "cardinality of S"

Example 2: $|\mathbb{N}| = |\mathbb{N} - \{1\}|$

Define a bijection from $\mathbb{N}$ to $(\mathbb{N} - \{1\})$.

$f : \mathbb{N} \to \mathbb{N}$

$f(n) = n+1$, is such a bijection. This is trivial from definition.

Similarly, if $k \in \mathbb{N}$,

$$|\mathbb{N}| = |\mathbb{N} - \{1, 2, 3, \ldots, k\}|.$$

Also, $|\mathbb{N}| = |2\mathbb{N}|$

($2\mathbb{N}$ = all even natural numbers)

by the map $f(n) = 2n$

Proposition $\mathbb{Z}$ is countable

proof: Define $\varphi : \mathbb{Z} \to \mathbb{N}$,

$$\varphi(n) = \begin{cases} 1, & n = 0 \\ 2n+1, & n > 0 \\ -2n, & n < 0 \end{cases}$$

I claim φ is a bijection.

injectivity: Suppose $n, m \in \mathbb{Z}$ and $\varphi(n) = \varphi(m)$.

cases i) $n = 0$. Then $\varphi(n) = 1$, so $\varphi(m) = 1$ Then $m \geq 0$ as $\varphi(k)$ is even $\forall\, k < 0$.

if $m > 0$, then

$\varphi(m) = 2m+1$. Hence,

$\varphi(m) \geq 3$. Therefore,

m must be equal to zero.

ii) $n < 0$, $m > 0$ then

$\varphi(n) = \varphi(m)$ is impossible

since $\varphi(n)$ is even and $\varphi(m)$

is odd.

(iii) $n, m < 0$ $\quad \varphi(n) = \varphi(m)$

$$-\cancel{2}n = -\cancel{2}m$$

$$n = m$$

(iv) $n, m > 0$ $\quad \varphi(n) = \varphi(m)$

$$2n+1 = 2m+1,$$

$$m = n.$$

This shows φ is injective.

Surjectivity show $\forall n \in \mathbb{N}$,

$\exists m \in \mathbb{Z}$, $\varphi(m) = n$.

If $n=1$, set $m=0$.

If n is even, set $m = -\frac{n}{2}$.

If n is odd, set $m = \frac{n-1}{2}$.
$(n>1)$

$\square$

Proposition: Any subset of a countable set is either finite, empty, or countable.

proof: Suppose $S \subseteq \mathbb{N}$ is infinite

By the well-ordering principle S has a least element t_1.

Define $f: \mathbb{N} \to S$

$$f(1) = t_1$$

Now consider $S - \{t_1\}$, This has a least element t_2. Define $f(2) = t_2$.

Inductively assume that we have defined f in this manner $\forall$ $k \in \mathbb{N}$, $1 \leq k \leq n \in \mathbb{N}$. Consider $S - \{t_1, t_2, \ldots, t_n\}$. This set has a least element t_{n+1}.

Define $f(n+1) = t_{n+1}$.

We must show f is bijective.

injectivity: Suppose $f(n) = f(m)$ for $n, m \in \mathbb{N}$. This means $f(n)$ is the smallest element in $S - \{t_1, \ldots, t_n\}$

and $f(m)$ is the smallest element in $S - \{t_1, \ldots, t_m\}$.

If $i, j \in \mathbb{N}$ and $i < j$, then $t_i < t_j$. This observation shows that if $n < m$, $t_n < t_m$ and if $n > m$, $t_n > t_m$

Therefore if $t_n = t_m$, $n = m$

Surjectivity: Let $m \in S$. Then $\exists$ only finitely many elements $s_1, s_2, \ldots, s_k$ for some $k \in \mathbb{N}$ with $s_i \in S$, $1 \leq i \leq k$.

By construction, $f(k+1) = m$.

$\square$

This proof is sufficient since any countable set is in bijection with $\mathbb{N}$.

Theorem. The countable union of countable sets is countable

proof: Suffices to show $\mathbb{N}$ contains a countable union of countable subsets.

Define $S_p \subseteq \mathbb{N}$, $S_p = \{p^n \mid n \in \mathbb{N}\}$ for p prime. Then if q is any other prime with $p \neq q$, then $S_p \cap S_q = \emptyset$ (from unique factorization).

$\{p \in \mathbb{N} \mid p \text{ is prime}\} \subseteq \mathbb{N}$

and is infinite, so by previous proposition, the set of all primes is countable.

$\bigcup_{p \text{ prime}} S_p \subsetneq \mathbb{N}$. $\bigcup_{p \text{ prime}} S_p$

is infinite since each S_p is infinite, so $\bigcup_{p \text{ prime}} S_p$ is

countable

In general, if $\{C_i\}_{i=1}^{\infty}$ is a disjoint countable collection of sets, define a surjection

$$\varphi : \bigcup_{i=1}^{\infty} C_i \to \bigcup_{p \text{ prime}} S_p$$

by enumerating the primes as $p_1, p_2, \ldots, p_n, \ldots$ and letting φ_i be a bijection from C_i to S_{p_i}. Extend to all of $\bigcup_{i=1}^{\infty} C_i$ by $\varphi(c) = \varphi_i(c)$ if $c \in C_i$. $\square$